G. General Solution to Angular Momentum Eigenvalue Problem

- Meaning: Rely only on $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, and $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$
  and their commutators

$$\left.\begin{aligned}
[\hat{L}_x, \hat{L}_y] &= i\hbar \hat{L}_z \; ; \quad [\hat{L}^2, \hat{L}_x] = 0 \\
[\hat{L}_y, \hat{L}_z] &= i\hbar \hat{L}_x \; ; \quad [\hat{L}^2, \hat{L}_y] = 0 \\
[\hat{L}_z, \hat{L}_x] &= i\hbar \hat{L}_y \; ; \quad [\hat{L}^2, \hat{L}_z] = 0
\end{aligned}\right\} \quad (1)$$

But not on any representation (e.g. spherical coordinates)

Know: Can find simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_z$ (one component)

Question: What can be said about the eigenstates and eigenvalues of $\hat{L}_z$ and $\hat{L}^2$ based entirely on commutation relations?

- To stress that we are considering <u>general</u> angular momentum, but <u>not only</u> orbital angular momentum, we use new symbols $(\hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}^2)$ for $(\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{L}^2)$

- <u>General Angular Momentum</u>

Eq. (1) [the commutation relations] <u>DEFINES</u> <u>angular momentum in QM</u>

<u>Meaning</u>: Whatever quantity that satisfies Eq. (1), it is called an angular momentum in QM

- Let ϕ be a simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$

$$\hat{J}^2 \phi = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{\alpha} \phi \quad \text{and} \quad \hat{J}_z \phi = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{\beta} \phi \qquad (2)$$

$\alpha \geq \beta^2$ (3) [component <u>cannot</u> be longer than length of vector]

- Introducing $\hat{J}_+$ and $\hat{J}_-$

$$\underbrace{\hat{J}_+ \equiv \hat{J}_x + i\hat{J}_y}_{\text{Raising Operator}} \quad ; \quad \underbrace{\hat{J}_- \equiv \hat{J}_x - i\hat{J}_y}_{\text{Lowering Operator}} \qquad (4)$$

[<u>Not</u> Hermitian] [<u>Not</u> Hermitian]

No problem! They do not represent physical quantities.

- $[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z \qquad (5)$

Why? $\hat{J}_+ \hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^2 + \hat{J}_y^2 - i \overbrace{[\hat{J}_x, \hat{J}_y]}^{i\hbar \hat{J}_z}$

$= \hat{J}_x^2 + \hat{J}_y^2 + \hbar \hat{J}_z$ (Useful later)

Similarly $\hat{J}_- \hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - \hbar \hat{J}_z$ (Useful later)

$$[\hat{J}_+, \hat{J}_-] = \hat{J}_+ \hat{J}_- - \hat{J}_- \hat{J}_+ = 2\hbar \hat{J}_z$$ Done!

- $[\hat{J}_z, \hat{J}_+] = [\hat{J}_z, \hat{J}_x + i\hat{J}_y] = [\hat{J}_z, \hat{J}_x] + i[\hat{J}_z, \hat{J}_y]$

$= i\hbar \hat{J}_y + i(-i\hbar \hat{J}_x)$

$= \hbar(\hat{J}_x + i\hat{J}_y) = \boxed{\hbar \hat{J}_+ = [\hat{J}_z, \hat{J}_+]} \quad (6)$

Similarly, $\boxed{[\hat{J}_z, \hat{J}_-] = -\hbar \hat{J}_-} \quad (7)$

- **What do $\hat{J}_+$ and $\hat{J}_-$ do?**

$$\hat{J}_z(\hat{J}_+\phi) = (\hat{J}_+\hat{J}_z + \hbar\hat{J}_+)\phi \qquad (\because [\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+)$$
$$= (\beta+\hbar)(\hat{J}_+\phi) \qquad (8)$$

$$\hat{J}_z(\hat{J}_-\phi) = (\hat{J}_-\hat{J}_z - \hbar\hat{J}_-)\phi \qquad (\because [\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_-)$$
$$= (\beta-\hbar)(\hat{J}_-\phi) \qquad (9)$$

Key Point

$\therefore$ If ϕ is eigenstate of $\hat{J}_z$ with eigenvalue β,

$(\hat{J}_+\phi)$ is also an eigenstate of $\hat{J}_z$ with eigenvalue $(\beta + \overbrace{\hbar}^{\text{one } \hbar \text{ higher}})$

$(\hat{J}_-\phi)$ is also an eigenstate of $\hat{J}_z$ with eigenvalue $(\beta - \underbrace{\hbar}_{\text{one } \hbar \text{ lower}})$ (10)

This is why $\hat{J}_+$ and $\hat{J}_-$ are called <u>raising</u> and <u>lowering</u> operators.

■ $\hat{J}^2\phi = \alpha\phi$. Will $\hat{J}_+$ and $\hat{J}_-$ change α?

$$[\hat{J}^2, \hat{J}_\pm] = [\hat{J}^2, \hat{J}_x \pm i\hat{J}_y] = \underbrace{[\hat{J}^2, \hat{J}_x]}_{0} \pm i \underbrace{[\hat{J}^2, \hat{J}_y]}_{0} = 0$$

$$\hat{J}^2(\hat{J}_+\phi) = \hat{J}_+\hat{J}^2\phi = \alpha(\hat{J}_+\phi) \qquad (11)$$

$$\hat{J}^2(\hat{J}_-\phi) = \hat{J}_-\hat{J}^2\phi = \alpha(\hat{J}_-\phi) \qquad (12)$$

$\therefore$ If ϕ is an eigenstate of $\hat{J}^2$ with eigenvalue α,
$(\hat{J}_+\phi)$ and $(\hat{J}_-\phi)$ are also eigenstates of $\hat{J}^2$ with eigenvalue α (13)

So, $\hat{J}_+$ and $\hat{J}_-$ respectively raise and lower eigenstates up or down the $\hat{J}_z$ eigenvalues in step of $\hbar$.

- Recall: $\alpha \geq \beta^2$ [component can't be longer than full length]

  For a particular value of α, there must be a maximum value of β (say β_{max}) and a minimum value of β (say β_{min})

  $\beta_{max} \leftrightarrow$ eigenstate $\phi_{\beta_{max}}$; $\beta_{min} \leftrightarrow$ eigenstate $\phi_{\beta_{min}}$

<u>Key ideas:</u>

$\hat{J}_+$ carries $\hat{J}_z$ eigenstate up in eigenvalue, but <u>there is a bound</u>

Eq.(8): $\hat{J}_z(\hat{J}_+\phi) = (\beta + \hbar)(\hat{J}_+\phi)$

- When we get to $\phi_{\beta_{max}}$, $\hat{J}_+\phi_{\beta_{max}}$ should <u>not</u> take it further higher
- But Eq.(8) must still remain true, i.e.

$$\boxed{\hat{J}_+\phi_{\beta_{max}} = 0} \quad (13)$$

$\left\{ \begin{array}{l} \text{defines } \phi_{\beta_{max}} \\ \text{gives an eq. for solving } \phi_{\beta_{max}}. \end{array} \right.$

$\hat{J}_-$ carries $\hat{J}_z$ eigenstate down in eigenvalue, but <u>there is a bound</u>

Eq.(9): $\hat{J}_z(\hat{J}_-\phi) = (\beta - \hbar)(\hat{J}_-\phi)$

- When we get to $\phi_{\beta_{min}}$, $\hat{J}_-\phi_{\beta_{min}}$ should <u>not</u> take it further lower
- But Eq.(9) must still remain true, i.e.

$$\boxed{\hat{J}_-\phi_{\beta_{min}} = 0} \qquad (14)$$

<u>Recall</u>:

$$\hat{J}_+\hat{J}_- = \hat{J}_x^2 + \hat{J}_y^2 + \hbar\hat{J}_z = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$$

$$\hat{J}_-\hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - \hbar\hat{J}_z = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$$

- $$\hat{J}_-\big(\underbrace{\hat{J}_+\phi_{\beta_{max}}}_{=0}\big) = \left(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z\right)\phi_{\beta_{max}} = 0$$

$$\Rightarrow \left(\alpha - \beta_{max}^2 - \hbar\beta_{max}\right)\phi_{\beta_{max}} = 0$$

$$\Rightarrow \boxed{\alpha = \beta_{max}(\beta_{max} + \hbar)} \qquad (15)$$

- $$\hat{J}_+\big(\underbrace{\hat{J}_-\phi_{\beta_{min}}}_{=0}\big) = \left(\hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z\right)\phi_{\beta_{min}} = 0$$

$$\Rightarrow \left(\alpha - \beta_{min}^2 + \hbar\beta_{min}\right)\phi_{\beta_{min}} = 0$$

$$\Rightarrow \boxed{\alpha = \beta_{min}(\beta_{min} - \hbar)} \qquad (16)$$

- α is the same in Eqs. (15) and (16): <u>β_{max} and β_{min} are related by</u>

$$\boxed{\beta_{min} = -\beta_{max}} \qquad (17) \quad \text{Key Result}$$

<u>Remark:</u> (17) looks familiar! E.g. d states in <u>orbital</u> angular momentum

Y_{22}, Y_{21}, Y_{20}, $Y_{2,-1}$, $Y_{2,-2}$, i.e. $\beta_{max} = +2\hbar$, $\beta_{min} = -\beta_{max} = -2\hbar$.

But, we need not be considering orbital AM here!

- Saw that $\hat{J}_z$ eigenvalues differ in step of $\hbar$

  [$\hat{J}_-$ lowers eigenvalue from β_{max} to $(\beta_{max} - \hbar)$, $(\beta_{max} - 2\hbar)$, ... β_{min}.]

$$\therefore \quad \beta_{max} - \beta_{min} = \text{integer} \times \hbar = n\hbar \quad [n \text{ is an integer}]$$

But $\beta_{min} = -\beta_{max}$, $\quad \beta_{max} - (-\beta_{max}) = 2\beta_{max} = n\hbar$

$$\Rightarrow \boxed{\beta_{max} = \frac{n}{2}\hbar} \qquad (18)$$

<u>Meaning:</u> Two cases

- n = even (e.g. $n=2$, J_z: $2\hbar, 1\hbar, 0, -\hbar, -2\hbar$)
- n = odd (e.g. $n=1$, J_z: $\frac{1}{2}\hbar, -\frac{1}{2}\hbar$)

- Write $j = \frac{n}{2}$ ($\therefore$ j is <u>either integer</u> OR <u>half-integer</u>)

$$\beta_{max} = j\hbar \; , \; \beta_{min} = -j\hbar \qquad [J_z = j\hbar, (j-1)\hbar, \cdots, -j\hbar]$$

Eq.(15) :

$$\underset{\text{eigenvalue of } \hat{J}^2}{\alpha} = \beta_{max}(\beta_{max} + \hbar) = j\hbar(j\hbar + \hbar) = \underbrace{j(j+1)\hbar^2}_{\text{must be of this form}}$$

<u>Conclusion:</u> Based on $\hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}^2$ commutation relations only, $\hat{J}^2$ has eigenvalues of the form $j(j+1)\hbar^2$, with j taking on integers <u>OR</u> half-integers.

For given value of j, $\hat{J}_z$ has eigenvalues that run from $j\hbar$ to $-j\hbar$ in steps of $\hbar$, giving $(2j+1)$ possible values.

Remarks

- The Most general statements about $\hat{J}^2$ and $\hat{J}_z$ eigenvalues
- Note: j can be $\underbrace{\text{half-integers}}$ OR $\underbrace{\text{integers}}$

e.g. $j=1/2$, $j=3/2, \cdots$ $j=0$, $j=1$, $j=2, \cdots$

$\underbrace{\text{[New]}}$ [Saw this in Orbital AM, $l=0$, $l=1$, $l=2$]

e.g. Electron has Spin Angular Momentum with $j=1/2$ (OR $s=1/2$)

- Why does l take on integers (but not half-integers)?
  - Orbital angular momentum $\rightarrow$ θ, ϕ in real space

$$\Phi(\phi) = \Phi(\phi+2\pi) \Rightarrow z\text{-components are } \underbrace{m_\ell}_{\text{integer!}} \hbar$$

- $j = 1/2$ case is particularly important (related to spin)

$$j = 1/2, \qquad J^2 = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2 \quad \text{OR} \quad J = \sqrt{\frac{3}{4}}\,\hbar$$

$$J_z = \underbrace{\frac{\hbar}{2},\ -\frac{\hbar}{2}}_{\text{2 possible values } \left[2\cdot\frac{1}{2}+1=2\right]}$$

<u>Summary</u>: Writing the results in a formal way

- $\hat{J}_z$: Eigenvalues written $m_j\hbar$ (introducing m_j)

  Thus, for given j, $m_j = j, j-1, \ldots, -j$

- A Simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ is defined (or <u>labelled</u>) by j and m_j [j gives $j(j+1)\hbar^2$ for $\hat{J}^2$ and m_j gives $m_j\hbar$ for $\hat{J}_z$]

- Use the symbol $|j\, m_j\rangle$ for the simultaneous eigenstate

$$\boxed{\begin{aligned} \therefore\ \hat{J}^2 |j\, m_j\rangle &= j(j+1)\hbar^2\, |j\, m_j\rangle \\ \hat{J}_z |j\, m_j\rangle &= m_j\hbar\, |j\, m_j\rangle \end{aligned}} \quad (19)$$

Given j, m_j goes from j to $-j$ in steps of 1

- Eq. (19) states the general results without referring to any way (representation) of expressing the AM operators AND the eigenstates
- In this notation, Eq. (10) says

$$\hat{J}_+ |j\, m_j\rangle \propto |j\, m_j+1\rangle \, ; \quad \hat{J}_- |j\, m_j\rangle \propto |j\, m_j-1\rangle$$

- See p. X-33, $\hat{J}_- \hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - \hbar \hat{J}_z = \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z$

$$\hat{J}_- \hat{J}_+ |j\, m_j\rangle = (\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z)|j\, m_j\rangle = [j(j+1) - m_j(m_j+1)]\hbar^2 |j\, m_j\rangle$$

using this notation

- Eq. (19) works for all angular momenta

Go To Ch. XI on <u>Spin Angular Momentum</u>